



Trapezoids and Deltoids in Wide Planar Point Sets

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Abstract

We call a set of n points in the Euclidean plane “wide” if at most \sqrt{n} of its points are collinear. We show that in such sets, the maximum possible number of trapezoids is $\Omega(n^3 \log n)$ and $O(n^3 \log^2 n)$ while for deltoids we have $\Omega(n^{5/2})$ and $O(n^{8/3} \log n)$.

Keywords Trapezoid · Deltoid · Order of magnitude

Introduction

Given a class \mathcal{C} of geometric objects in \mathbb{R}^2 , it is natural to ask the question: for a carefully selected set \mathcal{P} of n points, at most how many of the subsets of \mathcal{P} can be in \mathcal{C} ?

The first such question was posed by Erdős [2] on the number of unit distances (i.e., if \mathcal{C} is the class of all unit segments in \mathbb{R}^2) and it is still one of the most challenging problems in Combinatorial Geometry (if not “the” most challenging one).

The systematic study of triangles was initiated by Pach and Sharir [5] and it resulted in several nice results and new problems, see [1] or [3]. It is also natural to consider polygons with more than three vertices.

The goal of this paper is to study classes \mathcal{C} of quadrilaterals. Many of these types are easy to settle (or reduce to similar questions on triangles): squares, rectangles, rhombi, parallelograms and symmetric trapezoids. (A trapezoid is a quadrilateral with at least one pair of parallel sides.) However, two further classes produce interesting phenomena. For general, i.e., not necessarily symmetric trapezoids and also for deltoids (quadrilaterals whose four sides can be grouped into two pairs of adjacent equal-length sides), it makes a significant difference whether we allow arbitrary point sets or just wide (or “grid-like”) ones in the sense that we do not allow for more than \sqrt{n} collinear points. Our main results for this case (Theorems 1 and 2) can be summed up as follows.

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Theorem Consider all sets of n points in \mathbb{R}^2 such that at most \sqrt{n} of them are collinear. Then

- (a) the maximal number of trapezoids is $\Omega(n^3 \log n)$ and $O(n^3 \log^2 n)$;
- (b) the maximal number of deltoids is $\Omega(n^{5/2})$ and $O(n^{8/3} \log n)$.

Here and in what follows, for functions $f, g > 0$ we write $f(n) = O(g(n))$ if $f(n) \leq Bg(n)$ for a constant $B > 0$ and all n ; and $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$, i.e., if $f(n) \geq \beta g(n)$ for a constant $\beta > 0$ and all n .

Actually we shall also prove some more general results, e.g. we consider arbitrary bounds $\lambda(n) \leq n$ in place of \sqrt{n} above (see Theorems 1 and 2).

Known Results for Triangles and Quadrilaterals

Triangles

Proposition Among n points in the Euclidean plane, we have the following bounds on the maximum possible number of various triangles.

- (a) equilateral: $\Theta(n^2)$, e.g., in regular Δ -lattice.
- (b) right triangles (no shape prescribed): $\Theta(n^2 \log n)$ [5].
- (c) isosceles triangles (no shape prescribed): $\Omega(n^2 \sqrt{\log n})$ and $O(n^{2.136})$ [3].

Remarks on the Proposition.

- (a) In (a), the upper bound is obvious since two points cannot determine more than two equilateral triangles.
- (b) Pach–Sharir show an order of magnitude of $\Theta(n^2 \log n)$ for a dense set of angles in place of 90° .
- (c) A bound of $O(n^2 \lambda(n))$ would imply a lower bound of $\Omega(n/\lambda(n))$ for the number of distinct distances (for arbitrary $\lambda(n)$). That is why the former order of magnitude must be superquadratic, see [2] for an example with only $O(n/\sqrt{\log n})$ distinct distances.

Simple Quadrilaterals

Number of points that determine it	Quadrilateral	Bound(s)	Example or source
2	Square	$\Theta(n^2)$	\square lattice
“2.5”	Rectangle	$\Omega(n^2)$ $O(n^2 \log n)$	\square Lattice Proposition 1.(b)
“2.5”	Rhomb	$\Omega(n^2)$ $O(n^{2.136})$	\square Lattice Proposition 1.(c)
3	Parallelogram	$\Theta(n^3)$	\square Lattice
3	Symm. trapezoid	$\Theta(n^3)$	Regular n -gon

Trapezoids and Deltoids

Object	General bound	Example: points on few parallel lines	$\leq O(\sqrt{n})$ Pts collinear.	Example or source
Trapezoid	$\Theta(n^4)$	$n/2 + n/2$ pts on two parallel lines	$O(n^3 \log^2 n) \Omega(n^3 \log n)$	Theorem 1 Lemma 1.1
Deltoid	$\Theta(n^3)$	$n/3 + n/3 + n/3$ pts on three parallel lines	$O(n^{8/3} \log n) \Omega(n^{5/2})$	Theorem 2 \square lattice

Remark: It is not difficult to see that a $\sqrt{n} \times \sqrt{n}$ square lattice contains $cn^3 \log n$ trapezoids (see the forthcoming Lemma 1.1). Moreover, if we exclude lines with at least three points (i.e., we require that the point set be in general position) then a bound of Cn^3 is easy to demonstrate. This can even be attained, e.g., for the vertices of a regular n -gon. Moreover, for deltoids and in case of three parallel lines, $n/6 + 2n/3 + n/6$ points are better than the uniform distribution.

Our Theorems do not hold in finite affine planes: it is easy to see that the number of trapezoids is asymptotically as large as $n^{3.5}$ in such a plane of n points, with \sqrt{n} points/lines; while the number of deltoids (if defined appropriately) is $\sim n^3$. That is why we need some tool specific to the Euclidean plane. It is the following.

Theorem 0 (Szemerédi–Trotter) *For any n points of the Euclidean plane and for any $2 \leq k \leq n$, the number of straight lines with $\geq k$ points cannot exceed*

$$\begin{cases} C_0 \cdot \frac{n^2}{k^3}, & \text{if } k \leq \sqrt{n}; \\ C_0 \cdot \frac{n}{k}, & \text{if } k \geq \sqrt{n}; \end{cases}$$

where C_0 is an absolute constant (shown to be between 0.4 and 2.6 in [4]).

1 Trapezoids

Let $\lambda(n) \leq n$ be given, and assume that in a set of n points in the Euclidean plane, at most $\lambda(n)$ are collinear. Denote by $T(n)$ the maximum possible number of trapezoids (we prescribe no specific shape for them) in such sets.

Theorem 1

$$T(n) = \begin{cases} \Omega(n^3), & \text{for all } \lambda(n); \\ \Omega(n^3 \log n), & \text{if } \sqrt{n} \leq \lambda(n) \leq \sqrt{n} \log n; \\ O(n^3 \log^2 n), & \text{if } \lambda(n) \leq \sqrt{n} \log n; \\ \Theta(n^2 \lambda^2(n)), & \text{if } \sqrt{n} \log n \leq \lambda(n) \leq n. \end{cases}$$

The first lower bound is trivial: as mentioned before, a regular n -gon contains this many (even symmetric) trapezoids.

The second lower bound for $\sqrt{n} \leq \lambda(n) \leq \sqrt{n} \log n$ is stated as the following lemma.

Lemma 1.1 *A $\sqrt{n} \times \sqrt{n}$ square lattice contains $\Omega(n^3 \log n)$ trapezoids.*

Proof Given a $\sqrt{n} \times \sqrt{n}$ square lattice, we shall only estimate the number of those trapezoids whose parallel edges have positive slope i/j less than 1. (The total would be more than four times this quantity.) We shall call this fraction i/j the **slope of the trapezoid**. (If it is a parallelogram, we just pick one pair of parallel edges; at the end this will cost at most a factor of 1/2.) Moreover, in order to have several such trapezoids for j fixed, we only consider $j \leq \sqrt{n}/2$. Finally, to avoid multiple counting, we assume that i and j are coprime. In other words, for each pair (i, j) with $1 \leq i < j \leq \sqrt{n}/2$ and $\gcd(i, j) = 1$, we estimate the number of trapezoids of slope i/j .

In what follows, we shall use that $\lfloor x \rfloor \geq x/2$ if $x \geq 2$ (here $\lfloor x \rfloor$ is the lower integer part of x). E.g., for i, j fixed as above, there are at least

$$\left\lfloor \frac{\sqrt{n} \cdot j}{2} \right\rfloor \geq \frac{\sqrt{n} \cdot j}{4} \text{ lines of slope } i/j \text{ with } \left\lfloor \frac{\sqrt{n}}{j} \right\rfloor \geq \frac{\sqrt{n}}{2j} \text{ points on each.}$$

Of these, the number of possibilities to pick 2+2 vertices of a trapezoid of slope i/j is at least

$$\binom{\sqrt{n} \cdot j/4}{2} \cdot \binom{\sqrt{n}/2j}{2} \geq \frac{nj^2}{64} \left(\frac{n}{16j^2} \right)^2 = \Theta \left(\frac{n^3}{j^2} \right).$$

The number of $i < j$ coprime to j is given by Euler's function $\phi(j)$. Summing for all j in question, we get the required lower bound:

$$\Omega(n^3) \cdot \sum_{2 \leq j \leq \sqrt{n}/2} \frac{\phi(j)}{j^2} = \Omega(n^3 \log(\sqrt{n}/2)) = \Omega(n^3 \log n).$$

□

The third lower bound (hidden in the Θ notation) is, again, not difficult, not even for arbitrary $\lambda(n) \leq n$: draw $n/\lambda(n)$ equally spaced horizontal straight lines and place $\lambda(n)$ points on each, with all n horizontal coordinates independent over the rationals. This, on the one hand, guarantees that only horizontal triples can be collinear. On the other hand, the number of possibilities for first picking two of the lines and then two points on each is

$$\binom{n/\lambda(n)}{2} \cdot \binom{\lambda(n)}{2}^2 \approx \frac{1}{8} \cdot \frac{n^2}{\lambda^2(n)} \cdot \lambda^4(n) = \Omega(n^2 \lambda^2(n)),$$

as required. □

The upper bounds in Theorem 1 will be proven by showing an essentially equivalent incidence result. To have simple notations, we first apply a projection that maps the

line at infinity to a finite line \mathbf{e} while all n given points remain finite. In what follows we only work with these (projected, finite) points.

Let \mathcal{P} denote the set of these n points (note that $\mathbf{e} \cap \mathcal{P} = \emptyset$) and let $\{\ell_1, \ell_2, \dots\}$ be the at most $n(n-1)/2$ distinct straight lines that connect them (their actual number will be irrelevant). For each ℓ_i put

$$k_i := |\mathcal{P} \cap \ell_i|.$$

Moreover, for each $D \in \mathbf{e}$ we define

$$w(D) := \sum_{\substack{\ell_i, \ell_j \nmid D \\ i \neq j}} k_i^2 k_j^2,$$

as an obvious upper bound on the number of trapezoids whose parallel edges (before they were projected) had had common direction that, after the projection, corresponds to $D \in \mathbf{e}$. Actually, with $\binom{k_i}{2} \binom{k_j}{2}$ in place of $k_i^2 k_j^2$ we would get the exact count—but our bound w will be easier to use.

The upper bounds in Theorem 1 obviously follow from (actually they are equivalent to) the following.

Lemma 1.2 *If, in a set \mathcal{P} of n points, at most $\lambda(n)$ are collinear and $\mathbf{e} \cap \mathcal{P} = \emptyset$ then, using the foregoing notations,*

$$W(\mathbf{e}) := \sum_{D \in \mathbf{e}} w(D) \leq \begin{cases} O(n^3 \log^2 n), & \text{if } \lambda(n) \leq \sqrt{n} \log n; \\ O(n^2 \lambda^2(n)), & \text{if } \sqrt{n} \log n \leq \lambda(n) \leq n. \end{cases}$$

Proof For each $D \in \mathbf{e}$ and $2 \leq K \leq n$ put

$$n_D(K) := |\{\ell_i \nmid D; k_i = K\}|.$$

Using this notation we have

$$W(\mathbf{e}) = \sum_{D \in \mathbf{e}} w(D) = \sum_{D \in \mathbf{e}} \sum_{2 \leq K_1 \leq B\sqrt{n}} \sum_{2 \leq K_2 \leq B\sqrt{n}} n_D(K_1) n_D(K_2) \cdot K_1^2 K_2^2.$$

For a fixed pair $2 \leq K', K'' \leq n$, the contribution of the pairs (ℓ_i, ℓ_j) of lines with $K' \leq k_i < 2K', K'' \leq k_j < 2K''$ is

$$\begin{aligned} W_{K', K''}(\mathbf{e}) &:= \sum_{D \in \mathbf{e}} \sum_{K' \leq K_1 < 2K'} \sum_{K'' \leq K_2 < 2K''} n_D(K_1) n_D(K_2) \cdot K_1^2 K_2^2 \\ &\leq 16(K')^2 (K'')^2 \sum_{D \in \mathbf{e}} N_D(K') N_D(K''), \end{aligned}$$

where

$$N_D(K') := \sum_{K' \leq K_1 < 2K'} n_D(K_1) \quad \text{and} \quad N_D(K'') := \sum_{K'' \leq K_2 < 2K''} n_D(K_2).$$

By Cauchy–Schwartz, we have the upper bound

$$\begin{aligned} W_{K', K''}(\mathbf{e}) &\leq 16(K')^2(K'')^2 \sqrt{\sum_{D \in \mathbf{e}} N_D^2(K')} \cdot \sqrt{\sum_{D \in \mathbf{e}} N_D^2(K'')} \\ &\leq 16(K')^2(K'')^2 \sqrt{\max_D N_D(K') \cdot \sum_{D \in \mathbf{e}} N_D(K')} \cdot \sqrt{\max_D N_D(K'') \cdot \sum_{D \in \mathbf{e}} N_D(K'')}. \end{aligned} \quad (1)$$

Lemma 1.3 *The total contribution of the pairs (ℓ_i, ℓ_j) of lines with $k_i, k_j \leq \sqrt{n}$ is $O(n^3 \log^2 n)$.*

Proof We use inequality (1). On the one hand, for lines through D with at least K points each, $N_D(K) \leq n/K$ is obvious. On the other hand, for $K \leq \sqrt{n}$, by the Szemerédi–Trotter Theorem 0(i) (and since each line only intersects \mathbf{e} in one point), we have

$$\sum_{D \in \mathbf{e}} N_D(K) \leq C_0 \cdot \frac{n^2}{K^3}.$$

Thus our bound (1) becomes

$$\leq 16(K')^2(K'')^2 \sqrt{\frac{n}{K'} \cdot C_0 \cdot \frac{n^2}{(K')^3}} \cdot \sqrt{\frac{n}{K''} \cdot C_0 \cdot \frac{n^2}{(K'')^3}} = 16C_0 \cdot n^3,$$

Summing for the $\log^2(\sqrt{n}) \leq \log^2 n$ pairs $K', K'' = 1, 2, 4, \dots, 2^i, \dots \leq \sqrt{n}$, we get the required inequality. \square

Lemma 1.4 *If $\lambda(n) \geq \sqrt{n} \log n$ then the total contribution of the pairs (ℓ_i, ℓ_j) of lines with $\sqrt{n} \leq \max\{k_i, k_j\} \leq \lambda(n)$ is $O(n^2 \lambda^2(n))$.*

Proof We use inequality (1) again. $N_D(K) \leq n/K$ is still obvious. Now we must distinguish two cases.

(a) If $\sqrt{n} \leq K', K'' \leq \lambda(n)$ then we use (ii) of the Szemerédi–Trotter Theorem 0:

$$\sum_{D \in \mathbf{e}} N_D(K) \leq C_0 \cdot \frac{n}{K}.$$

Thus our bound (1) becomes

$$\leq 16(K')^2(K'')^2 \sqrt{\frac{n}{K'} \cdot C_0 \cdot \frac{n}{K'}} \cdot \sqrt{\frac{n}{K''} \cdot C_0 \cdot \frac{n}{K''}} = 16C_0 \cdot n^2 K' K''.$$

Summing first for $K' = 1, 2, 4, \dots, 2^i, \dots \leq \lambda(n)$, and then for the same values of K'' , we get the required bound.

- (b) If, say, $K' \leq \sqrt{n} \leq K'' \leq \lambda(n)$ (the symmetric case is equivalent), we use both parts (i) and (ii) of the Szemerédi–Trotter Theorem 0 once:

$$\leq 16(K')^2(K'')^2 \sqrt{\frac{n}{K'} \cdot C_0 \cdot \frac{n^2}{(K')^3}} \cdot \sqrt{\frac{n}{K''} \cdot C_0 \cdot \frac{n}{K''}} = 16C_0 \cdot n^{5/2} K''.$$

Summing first for $K'' = 1, 2, 4, \dots, 2^i, \dots \leq \lambda(n)$, and then for the $\log n$ values $K' = 1, 2, 4, \dots, 2^i, \dots \leq \sqrt{n}$, we get the bound $O(n^{5/2} \lambda(n) \log n)$ —smaller than required. \square

This also finishes the proof of Theorem 1.

2 Deltoids

Let $2 \leq \lambda(n) \leq n$ and assume that a set of n points in the plane contains $\leq \lambda(n)$ collinear points. Denote by $D(n)$ the maximum possible number of deltoids (we prescribe no specific shape for them) in such sets.

Theorem 2

$$D(n) = \begin{cases} \Omega(n^2 \lambda(n)), & \text{for all such } \lambda(n); \\ O(n^2 \lambda^2(n)), & \text{for } \lambda(n) \leq n^{1/3} \cdot \sqrt{\log n}; \\ O(n^{8/3} \log n), & \text{for } n^{1/3} \cdot \sqrt{\log n} \leq \lambda(n) \leq n^{2/3} \log n; \\ \Theta(n^2 \lambda(n)), & \text{for } n^{2/3} \log n \leq \lambda(n). \end{cases}$$

Proof To show the general lower bound, assume without loss of generality that $\lambda(n) \leq n/3$ and draw $m := \lceil n/\lambda(n) \rceil$ lines through the origin such that the angle between any consecutive pair is π/m . Place $\lambda(n)$ (or $\lambda(n) - 1$) points on one line such that the distances from the origin be algebraically independent transcendental numbers. Finally, rotate this point set repeatedly by $2\pi/m$ to copy the points to every other line. On the one hand, only triples of points from one line can be collinear by transcendental. On the other hand, each line can be the axis of at least

$$\binom{\lambda(n) - 1}{2} \cdot \frac{n - \lambda(n)}{2} \geq \frac{\lambda^2(n)}{8} \cdot \frac{n}{3},$$

deltoids, yielding a total of

$$m \cdot \frac{n \lambda^2(n)}{24} \geq \frac{n}{\lambda(n)} \cdot \frac{n \lambda^2(n)}{24} = \frac{n^2 \lambda(n)}{24}.$$

Remark (a) For $n = 3k$ and $\lambda(n) = k$, the configuration placed on three concurrent lines at 60° apart yields $3 \cdot \binom{k}{2} \cdot k \approx 3k^3/2 = n^3/18$ deltoids, which is better (by a factor of $3/2$) than the best distribution on three *parallel* lines.

- (b) For $\lambda(n) = \sqrt{n}$, this construction gives the same order of magnitude $n^{5/2}$ as a $\sqrt{n} \times \sqrt{n}$ square lattice (where a similar way of counting works).

As for a general upper bound we show the following.

Lemma 2.1 *For any $\lambda(n) \leq n$, we have $D(n) = O(n^2 \lambda^2(n))$.*

Indeed, every pair of points can form the axis of at most $\lambda^2(n)$ deltoids. \square

We are left to show the upper bounds for the last two cases of the Theorem. To this end, as usual, for every line ℓ that connects two points, we denote the number of points on ℓ by $k_\ell := |\mathcal{P} \cap \ell|$ and by σ_ℓ the number of pairs that are mapped to each other if we apply the reflection through ℓ . Then, in a maximal configuration, we have

$$D(n) = \sum_{\ell} \binom{k_{\ell}}{2} \cdot \sigma_{\ell} \leq \sum_{\ell} k_{\ell}^2 \cdot \sigma_{\ell}.$$

For a fixed pair (k, σ) , the contribution to $D(n)$ of those lines ℓ with $k \leq k_{\ell} < 2k$ and $\sigma \leq \sigma_{\ell} < 2\sigma$, is bounded by

$$D_{k,\sigma} := \sum_{\substack{k \leq k_{\ell} < 2k \\ \sigma \leq \sigma_{\ell} < 2\sigma}} k_{\ell}^2 \cdot \sigma_{\ell} \leq 8k^2 \cdot \sigma \cdot L(k, \sigma), \quad (2)$$

where $L(k, \sigma)$ denotes the number of lines in question.

Lemma 2.2 *The lines ℓ with $k_{\ell} \leq \sqrt{n}$ cannot contribute to $D(n)$ by more than $O(n^{8/3} \log n)$.*

Proof We have two bounds for $L(k, \sigma)$. On the one hand, it is $\leq n^2/\sigma$ since $\sum_{\ell} \sigma_{\ell} \leq n^2$ while, on the other hand, it is $\leq C_0 n^2/k^3$ by Theorem 0(i). Hence, in (2),

$$\begin{aligned} D_{k,\sigma} &\leq 8k^2 \cdot \sigma \cdot \min \left\{ C_0 \frac{n^2}{k^3}, \frac{n^2}{\sigma} \right\} = 8n^2 \cdot \min \left\{ C_0 \frac{\sigma}{k}, k^2 \right\} \\ &\leq 8n^2 (C_0 \sigma)^{2/3}, \end{aligned}$$

with equality in the last inequality iff $k = (C_0 \sigma)^{1/3}$. Summing for $\sigma = 1, 2, \dots, 2^i, \dots \leq n$, we get

$$\sum_{\sigma} D_{k,\sigma} = O(n^2) \cdot \sum_{2^i \leq n} (2^i)^{2/3} = O(n^2) \cdot O(n^{2/3}) = O(n^{8/3}).$$

Summing for the $\leq \log n$ values $k = 1, 2, 4, \dots, 2^i, \dots \leq \sqrt{n}$ we conclude that

$$D(n) = O(n^{8/3} \log n).$$

\square

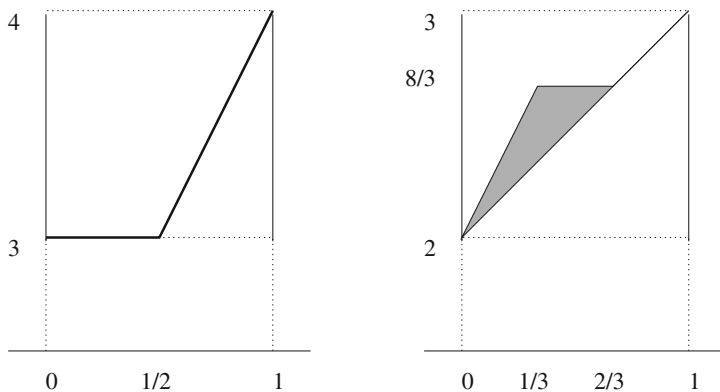


Fig. 1 For $\lambda(n) = n^x$ we have (a) $T(n) \sim n^y$ for trapezoids or (b) $D(n) \sim n^y$ for deltoids

Remark This order of magnitude could only be attained if we had “many”, more specifically, $\Omega(n)$ straight lines ℓ with $\sigma_\ell \sim n$ and $k_\ell \sim \sqrt[3]{n}$ simultaneously. However, this is unlikely since so many σ_ℓ being so large seems to force “many”, $\sim n$ concurrent lines—which contradicts $k_\ell \sim \sqrt[3]{n}$.

Lemma 2.3 *The lines ℓ with $\sqrt{n} \leq k_\ell \leq \lambda(n)$ can only contribute to $D(n)$ by at most $O(n^2 \lambda(n))$.*

Proof Again, we have two bounds for $L(k, \sigma)$. On the one hand, it is still $\leq n^2/\sigma$ while, on the other hand, we use part (ii) of Theorem 0 to get $L(k, \sigma) \leq C_0 n/k$. Hence, from (2),

$$D_{k,\sigma} \leq 8k^2\sigma \cdot \min \left\{ C_0 \frac{n}{k}, \frac{n^2}{\sigma} \right\} = 8nk \cdot \min\{C_0\sigma, nk\} \leq 8C_0nk\sigma = O(n^2\lambda(n)),$$

as required. □

This, finally, finishes the proof of Theorem 2, as well. □

Concluding Remarks

Apart from powers of $\log n$, the order of magnitude of $T(n)$, i.e. the exponent of n in $T(n)$ is quite well understood. This is not the case for deltoids where, for $\lambda \leq n^{2/3} \log n$, even the best possible exponent is unknown. (We believe that it must be closer to the lower bound, see the Remark after Lemma 2.2) (Fig. 1).

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References

1. Brass, P.: William Moser and János Pach. Research Problems in Discrete geometry. Springer, New York (2005)
2. Erdős, P.: On sets of distances of n points. Am. Math. Mon. **53**, 248–250 (1946)
3. Katz, N.H., Tardos, G.: A new entropy inequality for the Erdo's distance problem. In: Pach, J. (ed.) Towards a Theory of Geometric Graphs. Contemporary Mathematics, vol. 342, pp. 119–126. American Mathematical Society, Providence, RI (2004)
4. Pach, J., Radoičić, R., Tardos, G., Tóth, G.: Improving the crossing lemma by finding more crossings in sparse graphs. Discrete Comput. Geom. **36**(4), 527–552 (2006)
5. Pach, J., Sharir, M.: Repeated angles in the plane and related problems. J. Comb. Theory Ser. A **59**, 12–22 (1990)

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